On the Conversion of Orbital Elements from 1950.0 to J2000

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Abstract. Formulas are developed for the transformation of ecliptical orbital elements from B1950 to J2000. The results are compared with those recommended by IAU Commission 20. Some drawbacks to the Commission 20 formulation are pointed out and we develop procedures which are consistent with standard precessional formulations.

Key words: orbital elements, conversion from B1950 to J2000

1. Introduction

In 1990 IAU Commission 20 (West 1992) recommended some procedures for converting ecliptical orbital elements from the B1950 ecliptic and FK4 origin to the J2000 ecliptic and equinox. The procedures were developed by a committee headed by Yeomans and are contained in its *Report of the System Transition Committee* (Yeomans *et al.* 1990). The formulas also have been published in Chapter 5 of the new *Explanatory Supplement* (Standish *et al.* 1992). The relationships between the B1950 and J2000 orbital parameters as recommended by IAU Commission 20 are depicted in Fig. 1 where

$$\begin{aligned} L' &= 4^{\circ}\!.50001688 \\ L &= 5^{\circ}\!.19856209 \\ J &= 0^{\circ}\!.00651966. \end{aligned} \tag{1}$$

The difficulty with the Commission 20 formulation is that the values of L and L' differ by more than $0^{\circ}\!.5$ from what they would be if the equinox offset ΔE were zero, or if one would employ a formulation which is consistent with the established precessional formulations. In the above formulas it can be verified that

$$L - L' \approx p_A + \Delta E \cos \epsilon_{50}$$

where p_A is the accumulated general precession in longitude, $\Delta E = 0''\!.525$ is the "equinox offset" (dynamical equinox minus the FK4 origin) in equatorial coordinates, and where ϵ_{50} is the obliquity at 1950. The half-degree difference in L and L' is caused solely by the half-arcsecond value of ΔE and the manner in which the Commission 20 formulation is derived.

We will refer to the procedures recommended by Yeomans' IAU study group as the *Commission 20 recommendations*. The conversion of ecliptical orbital elements from unprimed B1950 values to primed J2000 values following the Commission 20 algorithm is given by Standish *et al.* (1992) as

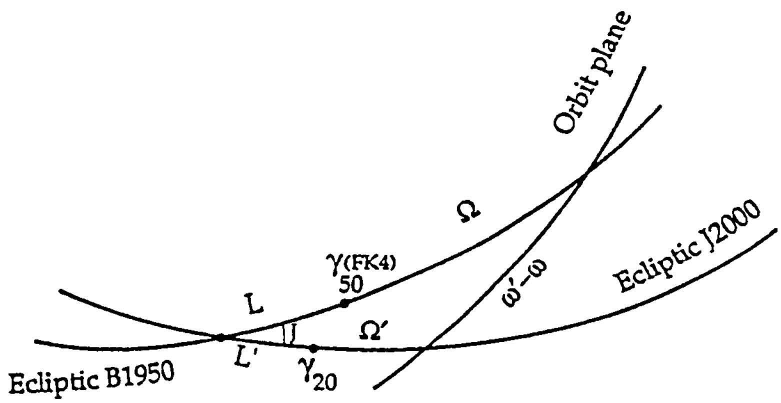

Fig. 1. Relationship between B1950 [unprimed quantities] and J2000 [primed quantities] orbital elements in IAU Commission 20 formulation.

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\begin{aligned}
\sin(\omega' - \omega)\sin I' &= \sin J \sin(L + \Omega) \\
\cos(\omega' - \omega)\sin I' &= \sin I \cos J + \cos I \sin J \cos(L + \Omega) \\
\cos I' &= \cos I \cos J - \sin I \sin J \cos(L + \Omega) \\
\sin(L' + \Omega')\sin I' &= \sin I \sin(L + \Omega) \\
\cos(L' + \Omega')\sin I' &= \cos I \sin J + \sin I \cos J \cos(L + \Omega)
\end{aligned} \tag{2}
$$

where ω represents the argument of periapsis, Ω the longitude of ascending node, and I the orbital inclination to the ecliptic.

The problem of interpretation becomes immediately apparent if one examines in more detail the intersection of the two ecliptic planes depicted in Fig. 1 and considers the other portion of the spherical triangle geometry depicted in Fig. 2. Well-established definitions (see *e.g.*, Andoyer 1911, or Lieske *et al.* 1977, Table 2), employed since the days of Bessel, describe the portion of the spherical triangles depicted in Fig. 2 as ecliptical precession parameters whose notation may vary from author to author but whose interpretation does not. In the case of the current IAU precession parameters, the notation of Lieske *et al.* (1977) is employed and the angles π_A, Π_A, and Λ_A are used, along with the accumulated general precession in longitude p_A.

2. An alternative formulation

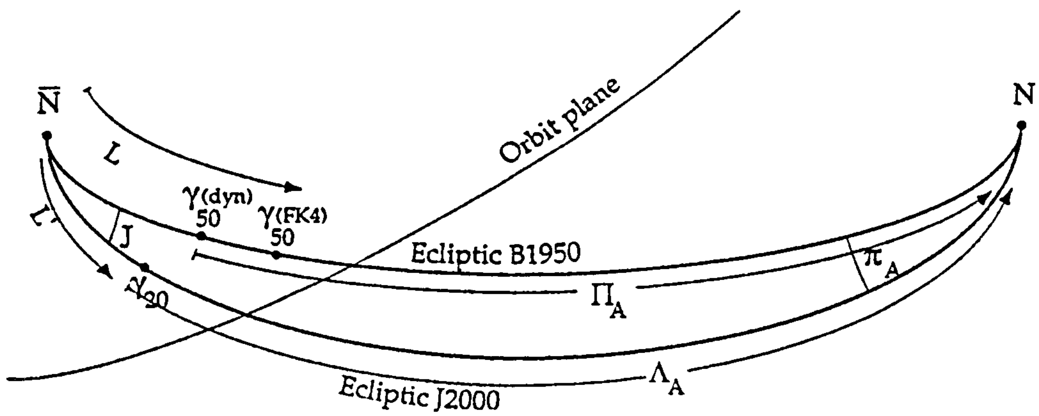

Fig. 2. Relationship between B1950 and J2000 ecliptic reference frames in IAU precessional formulation.

If we write the standard IAU precession parameters in two-argument form such as $f_A(t_A, t_B)$ where f_A represents some precessional parameter accumulated between epochs with Julian Dates t_A and t_B, then in Fig. 2 the arc from γ_{20} to N, measured along the J2000 ecliptic, could be written either as $\Lambda_A(B1950, J2000)$ or as $\Pi_A(J2000, B1950)$, where B1950 is the epoch for Besselian Year 1950 (JED 2433282.42345905) and where J2000 is the epoch for Julian Year 2000 (JED 2451545.0) [Lieske 1979]. Similarly, the angle J is $\pi_A(B1950, J2000)$ or $-\pi_A(J2000, B1950)$, if we employ the convention that $\mathrm{sgn}(\pi_A) = \mathrm{sgn}(t_B - t_A)$. The only possible ambiguity occurs in the arc from $\gamma_{50}(FK4)$ to N, measured along the 1950 ecliptic from the catalog origin of the FK4. It should be equal to $\Pi_A(B1950, J2000)$ or $\Lambda_A(J2000, B1950)$ minus a term whose size is on the order of the equinox offset in right ascension between the dynamical equinox and the FK4 origin, $\Delta E = 0\overset{''}{.}525$ (Fricke 1982), projected upon the ecliptic.

In the standard precessional formulation, $\Pi_A(B1950, J2000)$ represents the accumulated precessional displacement from the (*dynamical*) equinox of B1950, γ_{50}(dyn), to the intersection N of the J2000 ecliptic with that of 1950. Hence, the distance from the FK4 origin, $\gamma_{50}(FK4)$, to N should be $\Pi_A(B1950, J2000) - \Delta E \cos\epsilon_{50}$, where ϵ_{50} is the obliquity at B1950. That distance could also be written as $\Lambda_A(J2000, B1950) - \Delta E \cos\epsilon_{50}$.

In other words, we should have that

$$\begin{aligned} L' &= 180^\circ - \Lambda_A(B1950, J2000) \approx 5^\circ\!.0 \\ L &= 180^\circ - \Pi_A(B1950, J2000) + \Delta E \cos\epsilon_{50} \approx 5^\circ\!.7 \\ J &= \pi_A(B1950, J2000) \approx 0^\circ\!.0065, \end{aligned} \tag{3}$$

so that

$$\begin{aligned} L - L' &= \Lambda_A(B1950, J2000) - \Pi_A(B1950, J2000) + \Delta E \cos\epsilon_{50} \\ &= p_A(B1950, J2000) + \Delta E \cos\epsilon_{50}. \end{aligned} \tag{4}$$

In the standard IAU precession formulation (Lieske *et al.* 1977) the ecliptical precession parameters for displacements from epoch with Julian Ephemeris Date t_A to epoch t_B are

$$\begin{aligned} f_A(t_A, t_B) &= \phi_A(T, t) \\ \sin\pi_A \sin\Pi_A &= (4''\!.1976 - 0''\!.75250T + 0''\!.000431T^2)t \\ &\quad + (0''\!.19447 + 0''\!.000697T)t^2 - 0''\!.000179t^3 \\ \sin\pi_A \cos\Pi_A &= (-46''\!.8150 - 0''\!.00117T + 0''\!.005439T^2)t \\ &\quad + (0''\!.05059 - 0''\!.003712T)t^2 + 0''\!.000344t^3 \\ \Lambda_A - \Pi_A = p_A &= (5029''\!.0966 + 2''\!.22226T - 0''\!.000042T^2)t \\ &\quad + (1''\!.11113 - 0''\!.000042T)t^2 - 0''\!.000006t^3, \end{aligned} \tag{5}$$

where

$$\begin{aligned} T &= (t_A - 2451545.0)/36525 \\ t &= (t_B - t_A)/36525. \end{aligned} \tag{6}$$

By evaluating $\pi_A(B1950, J2000)$ and $\Pi_A(B1950, J2000)$ from the expressions for $\sin\pi_A \sin\Pi_A$ and $\sin\pi_A \cos\Pi_A$ [employing l'Hospital's rule to avoid the singularity at $t = 0$ for Π_A] in the standard IAU formulation and by evaluating $\Lambda_A(B1950, J2000)$ from

$$\Lambda_A = \Pi_A + p_A, \tag{7}$$

where p_A is the accumulated general precession in longitude, we find that

$$\begin{aligned} L' &= 5^\circ\!.00285 \\ L &= 5^\circ\!.70139 \\ J &= 0^\circ\!.00653. \end{aligned} \tag{8}$$

These values of L and L' differ from those of the Commission 20 recommendation by about $0^\circ\!.5$ –more than 3000 times greater than ΔE, which is on the order of $0''\!.5$. The important relationship between L and L' given by Eq. (4), however, is satisfied by the Commission 20 formulation.

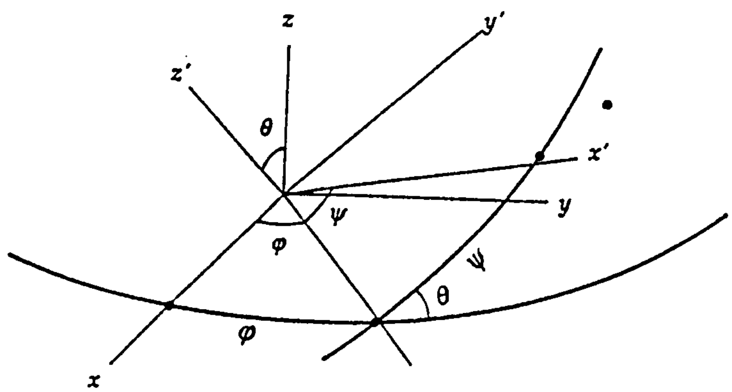

Fig. 3. Eulerian angles for the generalized rotation matrix $A = R_3(\psi)R_1(\theta)R_3(\phi)$ for use in $\mathbf{r}_{new} = A\mathbf{r}_{old}$.

3. Origin of the discrepancy

Let the elementary rotations be defined by

$$R_1(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}$$

$$R_2(\theta) = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \tag{9}$$

$$R_3(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

for rotations about the axes x, y, and z respectively. For a rotation about axis j through an angle α, measured positive in the counter-clockwise direction, the new coordinates $\mathbf{r}'$ are related to the old coordinates $\mathbf{r}$ by the formula

$$\mathbf{r}' = R_j(\alpha)\mathbf{r}.$$

Then as shown in Goldstein (1950) any combination of rotation matrices can be written as the product of three elementary rotations through the Eulerian angles

ϕ, θ and ψ depicted in Fig. 3:

$$\begin{aligned} A &= R_3(\psi)R_1(\theta)R_3(\phi) \\ \mathbf{r}' &= A\mathbf{r}. \end{aligned} \tag{10}$$

The Eulerian rotation angles can be derived from the components of A (where numerator and denominator are explicitly given in order that the arctan can be determined in the range from 0 to 360 deg):

$$\begin{aligned} \phi &= \arctan \frac{A_{3,1}}{-A_{3,2}} \\ \psi &= \arctan \frac{A_{1,3}}{A_{2,3}} \\ \theta &= \arccos A_{3,3}. \end{aligned} \tag{11}$$

The formulation employed in the Commission 20 recommendation is based upon consideration of the following matrix products (Standish 1992):

$$A = R_1(\epsilon_{20})P_{IAU}R_3(-\Delta E)R_1(-\epsilon_{50}) \tag{12}$$

where P_{IAU} is the standard IAU equatorial precession matrix of Lieske (1979),

$$\begin{aligned} P_{IAU} &= R_3(-90^\circ - z_A)R_1(\theta_A)R_3(90^\circ - \zeta_A) \\ &= R_3(-z_A)R_2(\theta_A)R_3(-\zeta_A). \end{aligned} \tag{13}$$

The sequence of rotations given in the Commission 20 formulation of Eq. (12) can be described as

1. a rotation R_1 [about the FK4 origin] by ϵ_{50} from the "ecliptic" of 1950.0 to the equator of B1950 ;
2. a rotation R_3 about the equatorial $z-$axis by ΔE from the $FK4$ origin in right ascension to the dynamical equinox;
3. a rotation by the standard IAU equatorial precession matrix from B1950 to J2000;
4. a rotation R_1 by ϵ_{20} from the equator of J2000 to the ecliptic of J2000.

When Eq. (12) is interpreted via Eq. (11), one finds the values of L, L' and J for the Commission 20 recommendation which are given in Eq. (1).

For comparison with the Commission 20 formulation, the proposed formulation given in Eq. (3) and Eq. (8) is equivalent to

$$A = R_1(\epsilon_{20})P_{IAU}R_1(-\epsilon_{50})R_3(-\Delta E \cos \epsilon_{50}), \tag{14}$$

although we actually employed the ecliptical precessional parameters so that

$$A = R_3(-\Lambda_A)R_1(\pi_A)R_3(\Pi_A)R_3(-\Delta E \cos \epsilon_{50}), \tag{15}$$

where all precessional displacements were taken from epoch B1950 to epoch J2000. In comparing with the Commission 20 formulation, the rotations employed in Eq. (14) can be described as

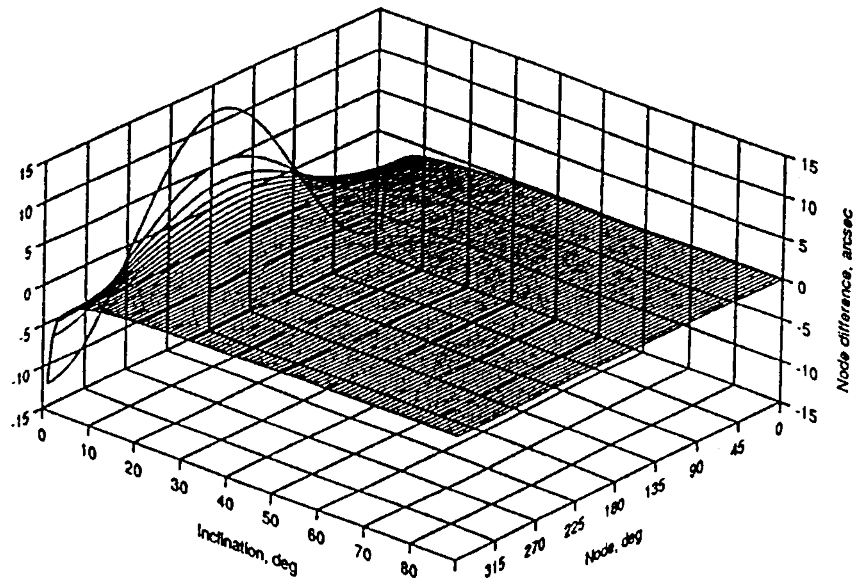

Fig. 4. Difference in longitude of ascending node on J2000 ecliptic between formulation of this paper and IAU Commission 20, $\Delta\Omega' = \Omega'(\text{thispaper}) - \Omega'(\text{Commission20})$, as a function of the 1950.0 inclination i and node Ω.

1. a rotation R_3 by $\Delta E \cos\epsilon_{50}$ from the $FK4$ origin in in ecliptical longitude to the dynamical equinox on the ecliptic of B1950;
2. a rotation R_1 by ϵ_{50} from the ecliptic of B1950 to the equator of B1950;
3. a rotation by the standard IAU equatorial precession matrix from B1950 to J2000;
4. a rotation R_1 by ϵ_{20} from the equator of J2000 to the ecliptic of J2000.

The different interpretation is due to another "ecliptic" being used in the first rotation $R_1(-\epsilon_{50})$ about the x-axis in the Commission 20 recommendation. The Commission 20 formulation makes the rotation about the FK4 origin rather than the dynamical equinox.

If $\mathbf{r}$ and $\dot{\mathbf{r}}$ are equatorial coordinates and velocities, then the orbital inclination and node relative to the equator are determined from their cross product. Based upon the value of $\mathbf{r} \times \dot{\mathbf{r}}$ and $\mathbf{r}' \times \dot{\mathbf{r}}'$ where

$$\mathbf{r}' = R_3(-\Delta E)\mathbf{r},$$

it can be shown that the effect of a rotation of $-\Delta E$ about the equatorial z-axis is to alter the longitude of ascending node on the equator by that amount. In other

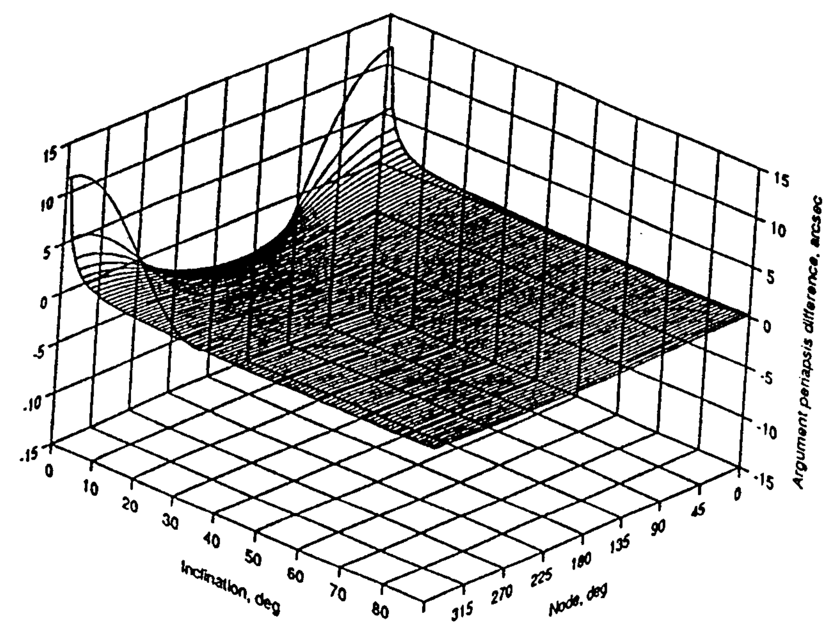

Fig. 5. Difference in argument of periapsis from J2000 ecliptic between formulation of this paper and IAU Commission 20, $\Delta\omega' = \omega'(\text{thispaper}) - \omega'(\text{Commission20})$, as a function of the 1950.0 inclination i and node Ω.

words, the plane of the orbit is not affected, but only its nodal description is.

Graphs of the differences in J2000 elements, derived from Eq. (2) using the formulation of Eq. (8) in the present paper and that of Commission 20 given in Eq. (1), are presented in Figs. 4 to 6. The difference in J2000 node $\Delta\Omega' = \Omega'(\text{thispaper}) - \Omega'(\text{Commission20})$ as a function of B1950 inclination i and node Ω is given in Fig. 4, while the difference for the argument of periapsis ω' is given in Fig. 5. As a reference value, the difference in longitude of periapsis $\varpi' = \omega' + \Omega'$ is given in Fig. 6 as a function of inclination and node. The comparison with longitude of periapsis is appropriate since the largest differences between the two formulations in node and argument of periapsis largely cancel each other.

4. Conclusion

The intent of the Commission 20 recommendation is to enable one to approximately convert orbital elements from the old 1950.0 system to the new J2000 system. In that sense, the current results are no "better" than the Commission 20 results and it probably would be best not to have a proliferation of alternatives such as has occurred in the conversion of star catalogue coordinates from 1950.0

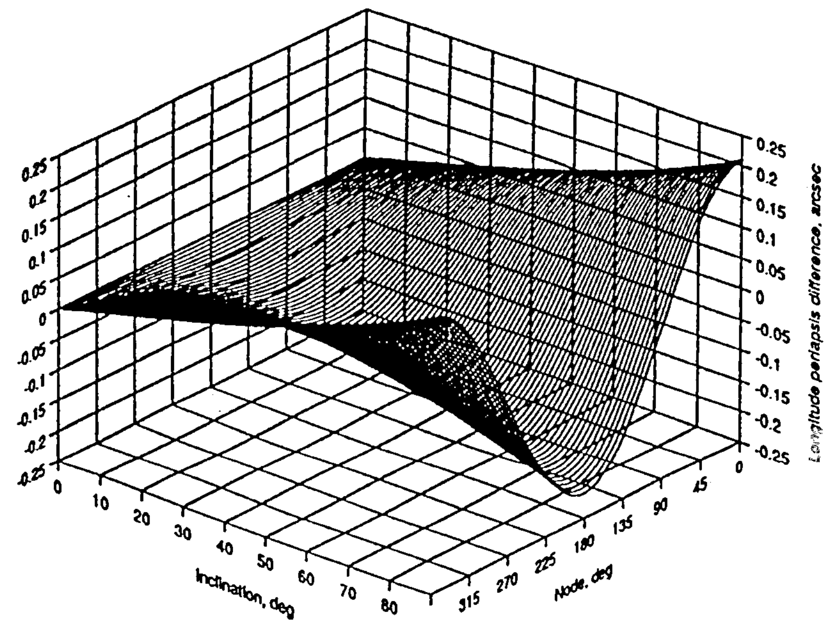

Fig. 6. Difference in longitude of periapsis ($\varpi = \omega + \Omega$) of J2000 ecliptic between formulation of this paper and IAU Commission 20, $\Delta\varpi' = \varpi'(\text{thispaper}) - \varpi'(\text{Commission20})$, as a function of the 1950.0 inclination i and node Ω.

to J2000 (*e.g.*, Standish 1982; Aoki *et al.* 1983; Lederle and Schwan 1984; Murray 1989; Yallop *et al.* 1989). However, given the possible confusion that might result from employing the Commission 20 parameters given in Eq. (1) with their known values from the standard precession formulation in ecliptical coordinates, the formulation of the present paper is recommended on the basis of consistency.

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